COHOMOLOGY RINGS OF PRECUBICAL SETS

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Abstract

The aim of this paper is to define the structure of a ring on a graded cohomology group of a precubical set in coefficients in a ring with unit.

Keywords: precubical cohomology rings, cohomology of small categories, precubical sets.

Introduction

Let G be the homologous system Abelian groups over a precubical set X [1], then for any integral $n \geqslant 0$, $H_n(X;G)$ are defined by values of satellites of the colimit functor $\varinjlim^n : \mathrm{Ab}^{(\square_+/X)^{op}} \to \mathrm{Ab}$, here $\square_+/X$ is a category of singular cubes of a precubical set X, Ab is the category of Abelian groups and homomorphisms, further for any small category $\mathscr{C}$ we denote by $\mathscr{C}^{op}$ the opposite category and finally $\mathrm{Ab}^{(\square_+/X)^{op}}$ is the category of functors from $(\square_+/X)^{op}$ to Ab. This observation is generalizing the Serre's spectral sequence for precubical sets [1]. For the cohomology groups there exist a opposite statement.

A cohomologous system over a precubical set we define as a functor on a category of singular cubes. In general, values of this functor on morphisms are not isomorphisms.

Suppose that the cohomologous system take constant values which are any ring R then we can to define a structure of a ring over a graded cohomology group with coefficients in this system.

The aim of this paper is to define the structure of a graded ring over a graded cohomology group of precubical sets with coefficients in the cohomologous system witch is taken a constant value. The basic result of this paper is Theorem 4.4.

We use following notations. The category of sets and maps we denote by Ens, Ab is the category of Abelian groups and homomorphisms and Ring is the category of rings and ring's homomorphisms which are save the unit.

1 Precubical Sets

Definition 1.1 *A precubical set $X = (X_n, \partial_i^{n,\varepsilon})$ is a sequence of sets $(X_n)_{n\in\mathbb{N}}$ with a fimile of maps $\partial_i^{n,\varepsilon} : X_n \to X_{n-1}$, defined for $i \leqslant i \leqslant n, \varepsilon \in \{0,1\}$, for which the following diagrams is commutative for all $\alpha, \beta \in \{0,1\}$, $n \geqslant 2$, $1 \leqslant i < j \leqslant n$:*

$$\begin{tikzcd}
Q_n \arrow[r, "\partial_j^{n,\beta}"] \arrow[d, "\partial_i^{n,\alpha}"'] & Q_{n-1} \arrow[d, "\partial_i^{n-1,\alpha}"] \\
Q_{n-1} \arrow[r, "\partial_{j-1}^{n-1,\beta}"'] & Q_{n-2}
\end{tikzcd}$$

Let $\square_+$ be a category consisting of finite sets $\mathbb{I}^n = \{0,1\}$ ordered as the Cartesian power of $\mathbb{I}$. Any morphism of the $\square_+$ is defined as an ascending map which admits a decomposition of the form $V_i^{k,\varepsilon} : \mathbb{I}^{k-1} \to \mathbb{I}^k$ where

$$V_i^{k,\varepsilon}(u_1, \dots, u_{k-1}) = (u_1, \dots, u_{i-1}, \varepsilon, u_i, \dots, u_{k-1}), \qquad \varepsilon \in \{0,1\}, \quad 0 \leqslant i \leqslant k.$$

here $\varepsilon \in \{0,1\}$, $1 \leqslant i \leqslant k$. Also we'll denote maps $V_i^{n,\varepsilon}$ by V_i^ε.

It well know [1] that any precubical set X is a functor $X : \square_+^{op} \to \mathrm{Ens}$.

Let H be a ordered subset $\{h_1, \dots, h_p\}$ of the set $\{1, 2, \dots, n\}$. Let us define a map $\lambda_H^\varepsilon : \mathbb{I}^p \to \mathbb{I}^n$ by the following formula

$$\lambda_h^\varepsilon \left(u_1, \dots, u_p\right) = \left(v_1, \dots, v_n\right),$$

where $v_i = \varepsilon$, if $i \notin H$, and $v_{h_r} = u_r$, $r = 1, \dots, p$.

Proposition 1.1 *Suppose that we have a subset $H = \{h_1, \dots, h_p\}$ of the set $\{1, 2, \dots, n\}$. Let us define following sets; $\widehat{H}_\mu = \{h_1, \dots, h_{\mu-1}, h_{\mu+1}, \dots, h_p\}$, $\widetilde{H}_\mu = \{h_1, \dots, h_{\mu-1}, h_{\mu+1}-1, \dots, h_p-1\}$. Further, let H_j be a $\{h_1, \dots, h_r, h_{r+1}-1, \dots, h_p - 1\}$ if $j \notin H$ and $h_r < j < h_{r+1}$. There are following formulas for $\varepsilon, \eta \in \{0,1\}$*

$$\lambda_H^\eta \circ V_\mu^\varepsilon = V_{h_\mu}^\varepsilon \circ \lambda_{\widetilde{H}_\mu}^\eta;$$

$$\lambda_H^\varepsilon \circ V_\mu^\varepsilon = \lambda_{\widehat{H}_\mu}^\varepsilon;$$

$$\lambda_H^\varepsilon = V_j^\varepsilon \circ \lambda_{H_j}^\varepsilon.$$

In this case, H_j and $\widetilde{H}_\mu$ are subsets of set $\{1, 2, \dots, n-1\}$.

Proof This proposition was proved in [2, Proposition 9.3.4]

2 A Diagonal Inclusion

In this section we'll introduce a diagonoal inclusion and show that this inclusion is the chain map.

First let us intruduce some notices from [1].

Let $X = (X_n, \partial_i^{n,\varepsilon})$ be the presubical set, let $\square_+[X_p] = L(X_p)$ for $p \geqslant 0$ be free Abelian group and $\square_+[X_p] = 0$ for $p < 0$. Assume that $D_i^\varepsilon = L(\partial_i^\varepsilon) : \square_+[X_p] \to \square_+[X_{p-1}]$. Further let us define homomorphisms

$$D : \square_+[X_p] \to \square_+[X_{p-1}], \quad p \geqslant 1,$$

by the formula

$$D = \sum_{i=1}^{p} (-1)^i \left(D_i^1 - D_i^0\right).$$

Let us assume that $\square_+[X] = \bigoplus\limits_{p \geqslant 0} \square_+[X_p]$ be the direct sum of groups $\square_+[X_p]$.

Following [1], identify cubes $f \in X_p$ with corresponding natural transformations $\widetilde{f} : h_{\mathbb{I}^p} \to X$ which are called *singular cubes*. Thus, singular p-cubes are elements of the group $\Box_+[X_p]$.

Let us consider functor morphisms $h_{\lambda^{\varepsilon}_H} : h_{\mathbb{I}^p} \to h_{\mathbb{I}^n}$, it's hard to see that the homomorphism D can define by the following corresponding $D^{\varepsilon} : f \mapsto f \circ h_{V^{\varepsilon}_i}$. It is clear that the $f \circ h_{V^{\varepsilon}_i}$ is define any face of the singular p-cube. There are rules of commutation functor morphisms $h_{\lambda^{\varepsilon}_H}$ with the homomorphism D in the following proposition which is a modification of proposition 1.1

Proposition 2.1 *Let us assume that we have a ordered subset* $G = \{g_1, \ldots, g_p\}$ *of set* $\{1, 2, \ldots, n\}$*. Suppose that* $\widehat{G}_\mu = \{g_1, \ldots, g_{\mu-1}, g_{\mu+1}, \ldots, g_p\}$ *and* $\widetilde{G}_\mu = \{g_1, \ldots, g_{\mu-1}, g_{\mu+1}-1, \ldots, g_p-1\}$*. Furhter, suppose that* $G_j = \{g_1, \ldots, g_r, g_{r+1}-1, \ldots, g_p-1\}$ *if* $j \notin G$ *and* $g_r < j < g_{r+1}$*. Let us assume that we have a pre-cubical set* $X = (X_n, \partial^{n,\varepsilon}_i)$*, let* $f : h_{\mathbb{I}^n} \to X$ *be a singular* n*-cube. There are following formulas for* $\varepsilon, \eta \in \{0, 1\}$ *:*

$$D^{\varepsilon}_{\mu}\left(f \circ h_{\lambda^{\eta}_G}\right) = D^{\varepsilon}_{g_\mu}(f) \circ h_{\lambda^{\eta}_{\widetilde{G}_\mu}} \tag{1}$$

$$D^{\varepsilon}_{\mu}\left(f \circ h_{\lambda^{\varepsilon}_G}\right) = f \circ h_{\lambda^{\varepsilon}_{\widehat{G}_\mu}} \tag{2}$$

$$f \circ h_{\lambda^{\varepsilon}_G} = D^{\varepsilon}_j(f) \circ h_{\lambda^{\varepsilon}_{G_j}} \tag{3}$$

We assumed that G_j *and* $\widetilde{G}_\mu$ *are ordered subsets of set* $\{1, 2, \ldots, n-1\}$*.*

Proof. From proposition 1.1 it follows that there are following formulas

$$h_{\lambda^{\eta}_G} \circ h_{V^{\varepsilon}_\mu} = h_{V^{\varepsilon}_{g_\mu}} \circ h_{\lambda^{\eta}_{\widetilde{G}_\mu}};$$

$$h_{\lambda^{\varepsilon}_G} \circ h_{V^{\varepsilon}_\mu} = h_{\lambda^{\varepsilon}_{\widehat{G}_\mu}};$$

$$h_{\lambda^{\varepsilon}_G} = h_{\lambda_{G^{\varepsilon}_j}} \circ h_{V^{\varepsilon}_j}.$$

Multiplying both sides by f, we complete the proof (see the commutative

diagramm).

$$\begin{tikzcd}
 & & X & & \\
h_{\mathbb{I}^{p-1}} \arrow[urr, bend left, "D^{\varepsilon}_{g_\mu}(f)\circ h_{\lambda^{\eta}_{\bar{G}_\mu}} = D^{\varepsilon}_{\mu}(f)\circ h_{\lambda^{\eta}_{G}}"] \arrow[rrr, "h_{V^{\varepsilon}_{\mu}}"] \arrow[dr, "h_{\lambda^{\eta}_{\bar{G}_\mu}}"'] & & & h_{\mathbb{I}^{p}} \arrow[ul, bend right, "f\circ h_{\lambda^{\eta}_{G}}"'] \arrow[dr, "h_{\lambda^{\eta}_{G}}"] & \\
 & h_{\mathbb{I}^{n-1}} \arrow[uur, "D^{\varepsilon}_{g_\mu}(f)"] \arrow[rrr, "h_{V^{\varepsilon}_{g_\mu}}"'] & & & h_{\mathbb{I}^{n}} \arrow[uull, bend right, "f"']
\end{tikzcd}$$

It well know (see [3]) that the tensor product $\square_+[X] \otimes \square_+[X]$ of the chain complex $\square_+[X]$ with itself is the chain complex $\square_+[X \otimes X]$, where

$$\square_+[(X \otimes X)_n] = \bigoplus_{p+q=n} \square_+[X_p] \otimes \square_+[X_q], \tag{4}$$

and bound operators is defined over generators $x \otimes x'$ by the formula

$$\partial(x \otimes x') = \partial x \otimes x' + (-1)^{\dim x} x \otimes \partial x'. \tag{5}$$

Proposition 2.2 *Let $X \in \square_+^{op}\mathrm{Ens}$ be a precubical set, let us assume that $\square_+[X]$ be aforesaid chain complex. Further let $\square_+[X \otimes X]$ be the tensor product of the chain complex $\square_+[X]$ with itself which defined by the formulas (4), (5). A map Δ (diagonal inclusion) which defined by the formula for any singular cub $f : h_{\mathbb{I}^n} \to X$:*

$$\Delta(f) = \sum_G \varrho_{GK} \left(f \circ h_{\lambda^0_G} \right) \otimes \left(f \circ h_{\lambda^1_K} \right),$$

is the chain map. Here K is the complement of a set $G = \{g_1, \ldots, g_p\} \subseteq \{1, 2, \ldots, n\}$, $varrho_{GK}$ is a signature of a permutation GK of integral numbers $1, 2, \ldots, n$. The summation is taken over all ordered subsets G of set $\{1, 2, \ldots n\}$.

Proof This poropsition was proved in [2, Proposition 9.3.5]

3 Cohomology of Precubical Sets with Coefficients in a Cohomologous System of Rings

Definition 3.1 *A cohomologous system of rings and a cohomologous system of Abelian groups over a precubical set* $X \in \square_+^{op}\mathrm{Ens}$ *are some functors* $\mathscr{R} : \square_+/X \to \mathrm{Ring}$ *and* $\mathscr{G} : \square_+/X \to \mathrm{Ab}$*, respectively.*

Let us consider Abelian grpups ${}^n\square_+[X,\mathscr{G}] = \prod\limits_{\vartheta\in X_n} \mathscr{G}(\vartheta)$. Let us define differentials $\delta_i^{n,\varepsilon} : {}^n\square_+[X,\mathscr{G}] \to {}^{n+1}\square_+[X,\mathscr{G}]$ as homomorphisms making following diagrams commutative

$$\begin{tikzcd}
\prod\limits_{\vartheta\in X_n} \mathscr{G}(\vartheta) \arrow[r, "\delta_i^{n,\varepsilon}"] \arrow[d, "\mathrm{pr}_{\vartheta\circ V_i^{n+1,\varepsilon}}"'] & \prod\limits_{\vartheta\in X_{n+1}} \mathscr{G}(\vartheta) \arrow[d, "\mathrm{pr}_\vartheta"] \\
\mathscr{G}\left(\vartheta \circ V_i^{n+1,\varepsilon}\right) \arrow[r, "\mathscr{G}\left(V_i^{n+1,\varepsilon} : \vartheta V_i^{n+1,\varepsilon} \to \vartheta\right)"'] & \mathscr{G}(\vartheta)
\end{tikzcd}$$

Definition 3.2 *Let* X *be a precubical set, let* $\mathscr{G} : \square_+/X \to \mathrm{Ab}$ *be a cohomologous system of Abelian groups over this precubical set* X*. A cohomology groups* $H^n(X;\mathscr{G})$ *of this precubical set* X *with coefficients in* $\mathscr{G}$ *are* n*-th cohomology groups of a chain complex* ${}^*\square_+[X,\mathscr{G}]$ *consisting of abelian groups*

$${}^n\square_+[X,\mathscr{G}] = \prod_{\sigma\in X_n} \mathscr{G}(\sigma)$$

and differentials

$$\delta^n = \sum_{i=1}^{n+1} (-1)^i (\delta_i^{n,1} - \delta_i^{n,0}).$$

Suppose that the cohomologous system of rings $\mathscr{R} : \square_+/X \to \mathrm{Ring}$ over a precubical set X take a constant value which is a ring R with a unity. Considering an additive component of the ring R we can examine a cohomology groups $H^*(X;R)$ with coefficient in the ring R.

Let ${}^*\square_+[X;R]$ be a cochain complex. Following [2, §5.7, 5.7.27] let us consider the homomorphism

$$\pi : {}^*\square_+[X;R] \otimes_R {}^*\square_+[X;R] \to {}^*\square_+[X\otimes X;R],$$

which defined by the formula

$$(\pi(u\otimes u'))\,(c\otimes c') = \eta\,(u(c)\otimes_R u'(c'))\,,$$

here $c, c' \in \square_+[X]$, $u, u' \in {}^*\square_+[X;R]$ and $\eta : R\otimes_R R \to R$ is an isomorphism of rings wich defined by the following formula

$$\eta\,(u(c)\otimes u'(c')) = u(c)\cdot u'(c'),$$

this prodoct is the multiplication operation in the ring R.

From [2, Proposition 5.7.28] follow that the homorphism π is the cochain map. Thus it's not hard to see that a map

$$\smile = \Delta^* \pi : {}^*\Box_+[X;R] \otimes_R {}^*\Box_+[X;R] \to {}^*\Box_+[X;R]$$

is the cochain map because from proposition 2.2 follows that the map Δ^* is the cochain map. It means that the $\smile$ generate some a product in $H^*(X;R)$. Thus we have the following

Theorem 3.1 *The graded group $H^*(X;R)$ with afore–mentioned $\smile$-product is a ring.*

Let us describe the $\smile$-product over cochains. Let $\varphi \in {}^p\Box_+[X;R]$ and $\psi \in {}^q\Box_+[X;R]$ are cochains. Let $u \in X_{p+q}$ be a $p+q$-cube. We have a formula

$$(\varphi \smile \psi)(u) = \sum_G \varrho_{GK} \varphi \left(u \circ h_{\lambda^0_G} \right) \cdot \psi \left(u \circ h_{\lambda^1_K} \right), \tag{6}$$

Here $G = \{g_1, \ldots, g_p\} \subseteq \{1, 2, \ldots, n\}$, ϱ_{GK} is a signature of a permutation GK of integral numbers $1, 2, \ldots, n$. The summation is taken over all ordered subsets G of set $\{1, 2, \ldots n\}$.

The notices of form $u \circ h_{\lambda^0_G}$ we also denote by $uh_{\lambda^0_G}$.

4 Properties of the Precubical Cohomology Ring

Here we will enumerate and we'll proof algebraic properties of the $\smile$-product in the ring $H^*(X;R)$.

Theorem 4.1 *The $\smile$-product of cochains in the ring ${}^*\Box_+[X;R]$ is associative and distributive with respect to the addition. If the ring R has left (right, two–sided) unit then the ring ${}^*\Box_+[X;R]$ has same unit.*

Proof. From associative and distributive of the product in the ring R follows associative and distributive of product in the ring ${}^*\Box_+[X;R]$. Further, let 1 — be a left unit of the ring R and let ι be a cochain which take each the 0-cube of the $\Box_+[X]$ to 1. It's not hard to see that for any cochain ξ there is the following equality $\iota \smile \xi = \xi$. In the same way we'll get the proof of this theorem if 1 is right or two–sided unit.

The cochain complex ${}^*\Box_+[X;R]$ with $\smile$-product is a graded ring.

Theorem 4.2 *For $\varphi \in {}^p\Box_+[X;R]$ $\psi \in {}^q\Box_+[X;R]$ there is the following formula*

$$\delta\left(\varphi \smile \psi\right) = \delta\varphi \smile \psi + (-1)^p \varphi \smile \delta\psi$$

Proof. We have

$$(\delta\varphi \smile \psi)(f) = \sum_G \varrho_{GK}\ (\delta\varphi)\left(fh_{\lambda^0_G}\right)\cdot\psi\left(fh_{\lambda^1_K}\right) =$$

$$= \sum_G \varrho_{GK}\left(\sum_{\mu=1}^{p+1}(-1)^\mu\left(\left(\delta^1_\mu\varphi\right)\left(fh_{\lambda^0_G}\right) - \left(\delta^0_\mu\varphi\right)\left(fh_{\lambda^0_G}\right)\right)\right)\cdot\psi\left(fh_{\lambda^1_K}\right),$$

$$(\varphi \smile \delta\psi)(f) = \sum_G \varrho_{GK}\ (\varphi)\left(fh_{\lambda^0_G}\right)\cdot(\delta\psi)\left(fh_{\lambda^1_K}\right) =$$

$$= \sum_G \varrho_{GK}\,\varphi\left(fh_{\lambda^0_G}\right)\cdot\left(\sum_{\eta=p}^{p+q+1}(-1)^\eta\left(\left(\delta^1_\eta\psi\right)\left(fh_{\lambda^1_K}\right) - \left(\delta^0_\eta\psi\right)\left(fh_{\lambda^1_K}\right)\right)\right),$$

here $G \subset \{1, 2, \ldots, p+q+1\}$, $G = (h_1, \ldots, h_{p+1})$ and K is the complement of the set G.

From the diagram

$$\begin{tikzcd}
h_{\mathbb{I}^{p+q+1}} \arrow[d, "f"'] & h_{\mathbb{I}^{p+1}} \arrow[l, "h_{\lambda^\xi_G}"'] \arrow[dl, "fh_{\lambda^\xi_G}"'] \\
X & h_{\mathbb{I}^p} \arrow[l, "D^\varepsilon_\mu\left(fh_{\lambda^\xi_G}\right)"] \arrow[u, "h_{V^\varepsilon_\mu}"']
\end{tikzcd}$$

and proposition 2.1 it follows that, we have

$$(\delta\varphi \smile \psi)(f) = \sum_G \varrho_{GK}\left(\sum_{\mu=1}^{p+1}(-1)^\mu\left(\varphi\left[D^1_{h_\mu}\left(fh_{\lambda^0_{\widehat{G}_\mu}}\right)\right]\right) - \varphi\left[fh_{\lambda^0_{\widehat{G}_\mu}}\right]\right)\cdot\psi\left[fh_{\lambda^1_K}\right],$$

$$(\varphi \smile \delta\psi)(f) = \sum_G \varrho_{GK}\,\varphi\left[fh_{\lambda^0_G}\right]\cdot\left(\sum_{\eta=p}^{p+q+1}(-1)^\eta\left(\psi\left[f\lambda^1_{\widehat{K}_\eta}\right]\right) - \psi\left[D^0_{k_\eta}\left(fh_{\lambda^1_{\widehat{K}_\eta}}\right)\right]\right).$$

Let $\check{K}_\mu$ be a complement of the set $\widehat{G}_\mu$. Let us consider a sum $(\delta\varphi \smile \psi)(f) + (-1)^p(\varphi \smile \delta\psi)(f)$. It's not hard to see that $\varphi\left[fh_{\lambda^0_{\widehat{G}_\mu}}\right]\cdot\psi\left[fh_{\lambda^1_K}\right]$ will appear twice; in the first place it will appear as a result of a deletion the g_μ from the G in the component (G, K) and in the second place it will appear as a result of a deletion the g_μ from the $\check{K}_\mu$ in the component $(\widehat{G}_\mu, \check{K}_\mu)$. In the first place $\varphi\left[fh_{\lambda^0_{\widehat{G}_\mu}}\right]\cdot\psi\left[fh_{\lambda^1_K}\right]$ hase a sign $\varrho_{GK}(-1)^{\mu+1}$, , further, in the second place it hase a sign $\varrho_{\widehat{G}_\mu\check{K}_\mu}(-1)^p(-1)^\alpha$, here $k_\alpha < g_\mu < k_{\alpha+1}$. But we have

$$\varrho_{\widehat{G}_\mu\check{K}_\mu} = (-1)^{p-\mu+\alpha}\varrho_{GK},$$

it means that the $\varphi\left[fh_{\lambda^0_{\widehat{G}_\mu}}\right]\cdot\psi\left[fh_{\lambda^1_K}\right]$ will appear twice with different signs. So that we have

$$(\delta\varphi\smile\psi)(f)+(-1)^p(\varphi\smile\delta\psi)(f)=$$

$$=\sum_G \varrho_{GK}\left(\sum_{\mu=1}^{p+1}(-1)^\mu\varphi\left[D^1_{g_\mu}\left(fh_{\lambda^0_{\widetilde{G}_\mu}}\right)\right]\cdot\psi\left[fh_{\lambda^1_K}\right]+\right.$$

$$\left.+(-1)^p\sum_{\eta=p}^{p+q+1}(-1)^{\eta+1}\varphi\left[fh_{\lambda^0_G}\right]\cdot\psi\left[D^0_{k_\eta}\left(fh_{\lambda^1_{\widetilde{K}_\eta}}\right)\right]\right). \tag{7}$$

From other side we have

$$(\delta(\varphi\smile\psi))(f)=\sum_{i=1}^{p+q+1}(-1)^i\left((\varphi\smile\psi)\left(D^1_if\right)-(\varphi\smile\psi)\left(D^0_if\right)\right)=$$

$$=\sum_{i=1}^{p+q+1}(-1)^i\sum_F\varrho_{FT}\left(\varphi\left[D^1_i(f)h_{\lambda^0_F}\right]\cdot\psi\left[D^1_i(f)h_{\lambda^1_T}\right]-\right.$$

$$\left.-\varphi\left[D^0_i(f)h_{\lambda^0_F}\right]\cdot\psi\left[D^0_i(f)h_{\lambda^1_T}\right]\right), \tag{8}$$

here F is an ordered subset of the set $\{1,2,\dots,p+q\}$ and T is its complement.

Using (1) – (3) of proposition 2.2, and assume that

$$F=\begin{cases}\widetilde{G}_j; & j\in G\\ G_j; & j\notin G\end{cases}\qquad\qquad T=\begin{cases}K_j; & j\in G\\ \widetilde{K}_j; & j\notin G\end{cases}$$

we get a bijection between triples (F,T,i) and (G,K,j) here $i=j$. It means that we have a bijection between (7) and (8) up to the sign. Let us prove that this signs are equal. We must check the following equation

$$(-1)^\mu\varrho_{GK}=(-1)^{h_\mu}\varrho_{\widetilde{G}_\mu K_\mu},\qquad(-1)^\eta\varrho_{GK}=(-1)^{k_\eta}\varrho_{G_\mu\widetilde{K}_\mu}.$$

Let us compare followings permutations

$$GK:\quad g_1,\dots,g_{\mu-1},g_\mu,\dots,g_p,k_1,\dots,k_\alpha,k_{\alpha+1},\dots,k_q$$

and

$$\widetilde{G}_\mu K_{h_\mu}:\quad g_1,\dots,g_{\mu-1},g_{\mu+1}-1,\dots,g_p-1,k_1,\dots,k_\alpha,k_{\alpha+1}-1,\dots,k_q-1,n.$$

It's not hard to see that following permutations

$$g_\mu,\dots,g_p,k_{\alpha+1},\dots,k_q\quad\text{and}\quad g_{\mu+1}-1,\dots,g_p-1,k_{\alpha+1}-1,\dots,k_q-1,n$$

have same signs, because we can get from first to second permutation by two steps: in the first step, we add 1 to all numbers, so we get $g_{\mu+1},\dots,g_p,k_{\alpha+1},\dots,k_q,g_\mu,$

and in the second step we transfer g_μ in the beginning. Each of this steps multiply the sing by $(-1)^{n-g_\mu}$. It means that sings of the last permutation are differents with respect to the $(-1)^\alpha$. Here α is a number of k which are smaller than g_μ, so that $\alpha = g_\mu - \mu$ and we complete to proof the first equation. In just the same way we can to proof the second equation.

Q.E.D.

Let a cochain complex is a graded ring with respect to any product, then this cochain complex is said [2] to be a *cochain ring*, if this product satisfy theorem 4.2. From theorem 4.2 we get the following

Corollary 4.3 *If φ and ψ are cocycles, then $\varphi \smile \psi$ is a cocycle. Moreover if ξ is a coboundary and ζ is a cocycle then $\xi \smile \zeta$ is a coboundary.*

Proof. Indeed, using theorem 4.2, we get

$$\delta(\varphi \smile \psi) = \delta(\varphi) \smile \psi + (-1)^{\dim \varphi} \varphi \smile (\delta\psi) = 0 + 0 = 0.$$

Let us suppose that $\xi = \delta\vartheta$ and let ξ be a coboundary, further let ζ be a cocycle, then

$$\delta(\vartheta \smile \zeta) = (\delta\vartheta) \smile \zeta + (-1)^{\dim \vartheta} \vartheta \smile (\delta\zeta) = \xi \smile \zeta.$$

This completes the proof of this Corollary.

Now we formulate the basic result of this paper.

Theorem 4.4 *A set $Z\,(X;R)$ of cocycles is a subring of the ring ${}^*\Box_+[X;R]$; a set $B\,(X;R)$ of coboundaries is a two–sided ideal in the ring $Z\,(X;R)$. The cohomology ring $H^*(X;R)$ of the a precubical set $X \in \Box_+^{op}\mathrm{Ens}$ is isomorphic to the quotient–ring $Z(X;R)/B(X;R)$. The ring $H^*(X;R)$ is a graded ring. If the ring R has left (right, two–sided) unity, then the ring $H^*(X;R)$ has the same unity.*

Proof. From Corollary 4.3 it follows that a set $Z(X;R)$ is a subring of the ring ${}^*\Box_+[X;R]$ and a set $B\,(X;R)$ is a two–sided ideal in the ring $Z\,(X;R)$. Further, from Definition 3.2 we get a additive isomorphism $H^*(X;R) \cong Z(X;R)/B(X;R)$. Suppose that $f, g \in H^*(X;R)$, let us consider their representatives $[f]$ and $[g]$ in $Z(X;R)$, respectively. It's not hard to see that using (6), we have that a representative of $f \smile g$ be $[f \smile g]$. It's evident that the above–cited cochain ι is a cocycle, this completes the proof of this Theorem.

Let us show that there is the following

Theorem 4.5 *If the ring R is a commutative then the ring $H^*(X;R)$ is an anticommutative.*

Proof. Since for any permutation GK of integral numbers $1, 2, \ldots, n$ there is the following equation $\varrho_{GK} = \varrho_{KG}$ then we get for any $\varphi \in {}^{p}\Box_{+}[X; R]$, $\psi \in {}^{q}\Box_{+}[X; R]$ the following equation

$$\varphi \smile \psi = (-1)^{pq} \psi \smile \varphi.$$

Q.E.D

Example 4.1 *Let us to calculate the cohomology ring of the torus* $\mathbb{T}^2$*. We present the torus* $\mathbb{T}^2$ *as a precubical set* $\mathbb{T}^2 = \left(Q_n\mathbb{T}^2; \partial_i^{n,\varepsilon}\right)$*, see the figure 1.*

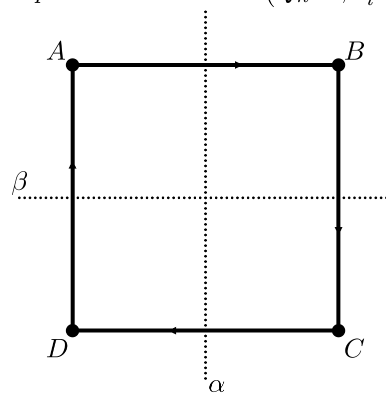

Figure 1: Here is shown the expanding of the torus; DA is identified with CB and AB is identified with DC.

So we have $Q_0\mathbb{T}^2 = \{o = A = B = C = D\}$, $Q_1\mathbb{T}^2 = \{t_1 = DA = CB, t_2 = AB = DC\}$, $Q_2\mathbb{T}^2 = \{\vartheta = ABCD\}$*. In the figure 2 are shown values which are taken bound differentials on the one and the two–dimension cubes.*

We have the following cochain complex

$$0 \to \mathbb{Z}^1 \xrightarrow{\delta^0} \mathbb{Z}^2 \xrightarrow{\delta^1} \mathbb{Z}^1 \xrightarrow{\delta^2} 0$$

Let us to assign k–dimension cochain ϑ^* to each k-cube ϑ of the precubical torus. This cochain ϑ^* is taken 1 on the cube ϑ and it is taken 0 on others cubes. We'll consider cochains which are the sum of cochains of form ϑ^*.

Since the following diagram is commutative

$$\begin{tikzcd}
f \in \prod\limits_{\vartheta \in Q_n\mathbb{T}^2} \mathbb{Z} \arrow[r, "\delta_i^{n,\varepsilon}"] \arrow[d, "\mathrm{pr}_{\vartheta \circ V_i^{n+1,\varepsilon}}"'] \arrow[dr, bend left, "\left(\delta_i^{n,\varepsilon} f\right)(\vartheta)"] \arrow[dr, bend right, "f\left(\vartheta V_i^{n+1,\varepsilon}\right)"'] & \prod\limits_{\vartheta \in Q_{n+1}\mathbb{T}^2} \mathbb{Z} \arrow[d, "\mathrm{pr}_{\vartheta}"] \\
\mathbb{Z} \arrow[r, "\mathbb{Z}\left(V_i^{n+1,\varepsilon} : \vartheta \circ V_i^{n+1,\varepsilon} \to \vartheta\right)"'] & \mathbb{Z}
\end{tikzcd}$$

then there exist the following equation

$$\left(\delta_i^{n,\varepsilon} f\right)(\vartheta) = f\left(\vartheta V_i^{n+1,\varepsilon}\right).$$

From this equation it's not hard to see that the one–dimension cochain f is taken different sign values on two edges of the bound of the 2-cube (according to the sign of the orientation of this 2-sube) then f is the cocycle.(see fig. **??***).*

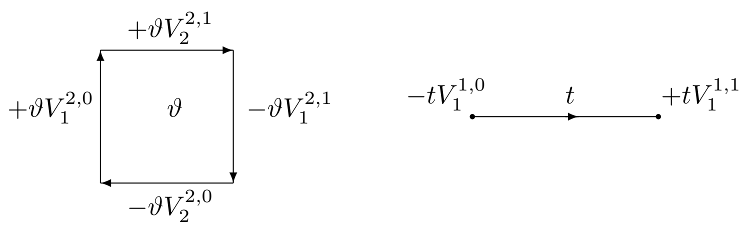

Figure 2: Here are shown the orientation of 2-cube and values of bound differentials $\vartheta V_i^{n,\varepsilon} = \partial_i^{n,\varepsilon}\vartheta$.

In figure 1, we have sketchy shown basic cocycles on the torus: if the dotted line is crossed any edge of the cube then the cocycle take 1 *on this edge, and this cocyle take* 0 *on others edges.*

Let us consider the $\smile$-product of basic cocycles. Since $\vartheta \circ V_i^{2,\varepsilon} = h_{\lambda^{\varepsilon}_{\{i\}}} \circ \vartheta$, we get (see (6) and figure 2.)

$$(\alpha \smile \beta)(\vartheta) = \alpha\left(\vartheta V_1^{2,0}\right)\cdot\beta\left(\vartheta V_2^{2,1}\right) - \alpha\left(\vartheta V_2^{2,0}\right)\cdot\beta\left(\vartheta V_1^{2,1}\right) = 0\cdot 0 - (-1)\cdot(-1) = -1.$$

Thus, $\beta \smile \alpha$ — is a basic cocycle of $H^2(\mathbb{T}^2;\mathbb{Z})$. Further

$$(\beta \smile \alpha)(\vartheta) = \beta\left(\vartheta V_1^{2,0}\right)\cdot\alpha\left(\vartheta V_2^{2,1}\right) - \beta\left(\vartheta V_2^{2,0}\right)\cdot\alpha\left(\vartheta V_1^{2,1}\right) = 1\cdot 1 - 0\cdot 0 = 1$$

So, we see that the cohomology ring $H^*(\mathbb{T}^2,\mathbb{Z})$ can be identified with the exterior algebra over the $\mathbb{Z}$-module $\mathbb{Z}$ whose generators are α and β.

Concluding Remark

So, let us to sum up. For any precubical set $X \in \square_+^{op}\mathrm{Ens}$ and for any ring R we get a graded cohomomology ring $H^*(X;R)$. If the ring R has the unit then the ring $H^*(X;R)$ has the same unit. Further, if the ring R is commutative then the ring $H^*(X;R)$ is anticommutative.

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